العلاقــة مابيــن الفضــاء T_1 والدالــة عديمــة النمــو

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ملخص البحث:

لنفرض أن X مجموعة ما و $g:X \leftarrow X$ دالة عديمة النمو في هذه الحالة نجد أن $g(X)=g_{fix}$ وإذا فرضنا أن $C(A) = A \cup g(A)$ لكل مجموعة A محتواة في المجموعة X فإن الدالة g يمكن أن تعرف تبولوجيا فوق X سوف نسميها بالتبولوجيا المتولدة بالدالة g ونرمز لها بالرمز τ_g. الآن إذا كان لدينا الثنائي (X,τ_g) عبارة عن فضاء T_1 سوف نبين أن g *يجب أن تكون دالة عديمة النمو والفضاء التبولوجي* X يجب أن يكون الفضاء التبولوجي المتقطع .

Relationship Between T_1-space and An Idempotent function

Abstract:

Let X be a set and $g:X \rightarrow X$ be an idempotent function. In this case we have $g(X)=g_{fix}$, and if we defined $C(A) = A \cup g(A)$ for all $A \subseteq X$ then g can determines a topology in a set X we will call it a topology induced by a function g and denote to it by τ_g . Now if (X, τ_g) is a T_1-space we will show that g must be an idempotent map and X must be a discrete topological space.

Key words: An idempotent function , discrete space , T_1-space.

Introduction:

If we have a finite T_1-space (X,τ), then (X,τ) must be a discrete topological space, because if we suppose that (X,τ) is an any finite T_1-space.Then by [Theorem 1-4] every single subset $\{x\}$ of X is closed in (X,τ).Since X is finite, it follows that any subset of X is closed, since it is a finite union of closed single sets. Thus any subset of X, as a complement of a closed set, is open in X, and hence X is a discrete space. But now the equation is that: Can we get the same resale if X is an any set (may it is not finite set)?. Theorem 2-7 will show that for any X, the pair (X, τ_g) is a T_1-space where g is an idempotent function ,and τ_g is a topology induced by the map g.

1- Basic Concepts:

Definition 1-1

Let X be any set, then a function $g : X \to X$ is said to be an idempotent function if : $g^2 = g \circ g = g$.

Example 1-2

Let $X = \{a, b, c\}$ and let g: $X \to X$ defined by:

$$g(x) = \begin{cases} x & if \quad x \neq b \\ a & if \quad x = b \end{cases} \qquad for\ all\ x \in X.$$

Then $g(g(a)) = a = g(a)$,
$g(g(b)) = a = g(b)$,
and $g(g(c)) = c = g(c)$.

Therefore $g^2(x) = g(x)$ for all $x \in X$, and hence g is an idempotent function .

Definition 1-3

A topological space X is a T_1-space if and only if when ever x and y are distinct points in X, there is a neighborhood of each not containing the other.

Theorem 1-4 [2]

Let X be any topological space then the following are equivalent :

(1) X is T_1-space .

(2) $\{x\}$ is closed for all $x \in X$.

(3) For any $A \subset X$, $A = \cap\{U : U \text{ is open }, A \subset U\}$.

Theorem 1-5 [2]

Given a set X and any family Ψ of subsets of X satisfying the conditions :

(1) Any intersection of members of Ψ belongs to Ψ.

(2) Any finite union of members of Ψ belongs to Ψ .

(3) Φ and X both belong to Ψ .

Then the collection of complements of members of Ψ is a topology on X in which the family of closed sets is just Ψ .

Theorem 1-6 [2]

If we have a set X and a mapping $g : P(X) \to P(X)$ satisfying the conditions :

(a) $A \subset g(A)$ for any $A \subset X$.

(b) $g(g(A)) = g(A)$ for any $A \subset X$.

(c) $g(A \cup B) = g(A) \cup g(B)$ for any $A, B \subset X$.

(d) $g(\Phi) = \Phi$.

Then g defines a topology on X in which the closure of A in X is $g(A)$, and the closure operation is just g.

2- Topologies induced by an idempotent functions:

Definition 2-1

Let g be an idempotent function on a set X . Then we define :

(1) $g_{fix} = \{x \in X : g(x) = x\}$

(2) $C(A) = A \cup g(A)$ for all $A \subseteq X$.

Theorem 2-2 [3]

Let $g : X \rightarrow X$ be an idempotent function . Then the operation
$C : P(X) \rightarrow P(X)$ defined by $C(A) = A \cup g(A)$ for all $A \subset X$ is a topological closure operation in the set X .

Definition 2-3

Let $g : X \rightarrow X$ be an idempotent function, and let $C : P(X) \rightarrow P(X)$ defined by $C(A) = A \cup g(A)$ for each $A \subset X$ and let $\Psi = \{C(A) : A \subset X\}$. Then the topology $\tau_g = \{F^c : F \in \Psi\}$ is called the topology induced by the map g.

Example 2-4

Let X be any set and let $i : P(X) \rightarrow P(X)$ be the identity map then i is an idempotent map and the topology induced by i is the discrete topology .

Theorem 2-5 [3]

Let $g : X \rightarrow X$ be an idempotent map then :

(a) $g_{fix} = g(X)$.

(b) For every element x of X the one element set $\{x\}$ is closed in the topology induced by g if and only if $x \in g_{fix}$.

Theorem 2-6

The frontier of any one element set in (X, τ_g) where τ_g is the topology induced by an idempotent map g is either empty or a one element set.

Proof:

Suppose that (X, τ_g) is a topological space , where τ_g is the topology induced by an idempotent map g , and suppose $Fr(\{x\})$; that is (frontier of $\{x\}$) is not empty and not a one element set for some $x \in X$. Let $y, y' \in Fr(\{x\})$. Where $y \neq y'$, so $y, y' \in C(\{x\}) \cap C(X \setminus \{x\}) = [\{x\} \cup g(\{x\})] \cap [(X \setminus$

$\{x\}) \cup g(X \setminus \{x\})]$, so $y, y' \in \{x\} \cup g(\{x\})$, and $y, y' \in (X \setminus \{x\}) \cup g(X \setminus \{x\})$ $\rightarrow$ (I)

So we have two cases :

Case (1)

If $y \neq x$, $y' \neq x$, then by (I) we have y , $y' \in g(x)$. So $g(x) = y$, and $g(x) = y'$, where $y \neq y'$ which is a contradiction.

Case (2)

If $x = y$ or $x = y'$. Suppose $x = y'$, then $x \neq y$. So by (I) we have $x, y \in \{x\} \cup g(x)$, so $y \in g(x)$, and hence $g(x) = y$. Since $x, y \in (X \setminus \{x\}) \cup g(X \setminus \{x\})$, $x \notin (X \setminus \{x\})$, so $x \in g(X \setminus \{x\})$, and there exists $z \in (X \setminus \{x\})$ such that $g(z) = x$. Now $(g \circ g)(z) = g(g(z)) = g(x) = y$ that is $(g \circ g)(z) = y \neq g(z) = x$ which is a contradiction .So $Fr(\{x\})$ is either empty or a one element set for any $x \in X$.

Theorem 2-7

If (X, τ_g) is a T_1-space , then g is the identity map and τ_g is the discrete topology .

Proof:

Let (X, τ_g) be a T_1-space , then by [Theorem 1-4] every one element set $\{x\}$ is closed , so $C(\{x\}) = \{x\}$ for all $x \in X$.

But $C(\{x\}) = \{x\} \cup g(\{x\}) = \{x\}$; that is $g(\{x\}) = \{x\}$. Therefore $x \in g_{fix}$ for all $x \in X$, and since $g(x) = x$ for all $x \in X$, so g is the identity map. Now we need to prove that τ_g is the discrete topology on X , since $g(A) = A$ for all $A \subseteq X$, and since $C(A) = A \cup g(A) = A \cup A = A$, so any subset of X is a closed set , and for any $x \in X$ we have $A = X \setminus \{x\}$ is a closed set, so $\{x\}$ is an open set in X , and hence τ_g is the discrete topology .

References:

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